MATH 121, Calculus I — Exam I (Fall 2013)

Name: ______________________________

KU ID No.: ______________________________

Lab Instructor: ______________________________

This exam has a total value of 200 points. It consists of two parts. The first part contains 14 multiple-choice questions, each worth 10 points. The second part contains 3 long-answer problems, each worth 20 points. There are 17 problems in total to be solved. Additionally the last page of the exam contains an extra-credit problem that is worth 20 points. This is strictly a closed-book exam and the use of technology (including calculators, phones, tablets, and laptops) is prohibited. No extra paper is allowed and only the work shown on the front side of each provided page of the exam will be graded.

Score

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7	
Problem 8	
Problem 9	
Problem 10	
Problem 11	
Problem 12	
Problem 13	
Problem 14	
Problem 15	
Problem 16	
Problem 17	
Extra Credit	
Total	

Honor code: "I have not cheated on this exam and I am not aware that anyone else has cheated on this exam."

Signatures: ______________________________ (at time of submission).

______________________________ (upon receipt of graded exam).

Multiple-Choice Questions

Instructions: Place the appropriate letter for your answer for each problem in the blank box that is provided. Correct answers do not require work to receive full credit. However, partial credit can be awarded to incorrect answers based on work shown in the adjacent blank space. Hence, you are strongly advised to show work for each problem.

1. [**10 points**] Find the exact value of $\lim_{x\to 0} \dfrac{\sqrt{3+x}-\sqrt{3}}{x}$.

   (A) $\sqrt{3}$

   (B) 0

   (C) $\dfrac{1}{2\sqrt{3}}$

   (D) The limit does not exist.

   Answer: C

$$= \lim_{x\to 0} \frac{(\sqrt{3+x}-\sqrt{3})(\sqrt{3+x}+\sqrt{3})}{x(\sqrt{3+x}+\sqrt{3})}$$

$$= \lim_{x\to 0} \frac{\cancel{3}+x-\cancel{3}}{x(\sqrt{3+x}+\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

or

$$\underset{H}{=} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{3+x}} - 0}{1} = \frac{1}{2\sqrt{3}}$$

or

$$= f'(3) \text{ where } f(x)=\sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \text{ and } f'(3) = \frac{1}{2\sqrt{3}}$$

2. [**10 points**] Let $f(x) = \frac{5}{2}x^2 - e^x$. Find the value of x for which the second derivative $f''(x)$ equals zero.

   (A) $\ln 5$

   (B) $5e$

   (C) 0

   (D) e^5

   Answer: A

$$f'(x) = 5x - e^x$$

$$f''(x) = 5 - e^x$$

$$f''(x) = 0 \text{ iff } 5 - e^x = 0$$

$$\text{iff } e^x = 5$$

$$\text{iff } \boxed{x = \ln 5}$$

3. **[10 points]** Find the derivative of $f(x) = \left(1 + x^4 - \frac{1}{x}\right)^{5/3}$.

(A) $\frac{20}{3}x^{17/3} + \frac{5}{3x^{8/3}}$

(B) $\frac{5}{3}\left(1 + x^4 + \frac{1}{x}\right)^{2/3}$

(C) $\frac{5}{3}\left(1 + x^4 - \frac{1}{x}\right)^{2/3}\left(4x^3 + \frac{1}{x^2}\right)$

(D) $\frac{5}{3}x^{2/3}\left(4x^3 + \frac{1}{x^2}\right)$

$$f'(x) = \frac{5}{3}\left(1+x^4-\frac{1}{x}\right)^{\frac{5}{3}-1} \cdot \left(1+x^4-\frac{1}{x}\right)'$$

$$= \underline{\frac{5}{3}\left(1+x^4-\frac{1}{x}\right)^{2/3} \cdot \left(4x^3+\frac{1}{x^2}\right)}$$

where $\left(-\frac{1}{x}\right)' = \left(-x^{-1}\right)' = +x^{-2} = \frac{1}{x^2}$

Answer: C

4. **[10 points]** Use implicit differentiation to find an equation of the tangent line to the curve

$$\sin x + \cos y = 1$$

at the point $(\pi/2, \pi/2)$.

(A) $y - \frac{\pi}{2} = 4\left(x - \frac{\pi}{2}\right)$

(B) $y = \pi$

(C) $y - \frac{\pi}{2} = \left(x - \frac{\pi}{2}\right)$

(D) $y = \frac{\pi}{2}$

Answer: $\frac{\pi}{2}$

$$\cos x - (\sin y) \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

$$\left.\frac{dy}{dx}\right|_{(\pi/2, \pi/2)} = \frac{\cos \pi/2}{\sin \pi/2} = 0$$

$$\Rightarrow y - \frac{\pi}{2} = 0 \cdot \left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow \boxed{y = \frac{\pi}{2}}$$

5. [**10 points**] Which of the following statements are true? (Since there may be more than one correct answer, determine all correct answers.)

(A) If $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists, then f is differentiable at a.

(B) If f is continuous at a, then f is differentiable at a.

(C) If $\lim_{x \to a} f(x)$ exists, then f is differentiable at a.

(D) If f is differentiable at a, then $\lim_{x \to a} f(x) = f(a)$.

Answer: A, D

6. [**10 points**] If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, $g'(5) = 6$, find $F'(5)$.

(A) 24

(B) 8

(C) 12

(D) 20

Answer: A

$F'(x) = f'(g(x)) \cdot g'(x)$

$F'(5) = f'(g(5)) \cdot g'(5)$

$F'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = \underline{24}$

7. [**10 points**] Find the domain of the function $f(x) = \sqrt{1-x}\,\ln x$.

(A) $(0, +\infty)$

(C) $(0, 1]$

(B) $(-\infty, 1)$

(D) $[0, 1]$

Answer: C

$D_f = \{x : 1-x \geq 0 \text{ and } x > 0\}$

$= \{x : x \leq 1 \text{ and } x > 0\}$

or $D_f = (0, 1]$

8. [**10 points**] For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x, & \text{if } x < 2; \\ 2x + 4, & \text{if } x \geq 2. \end{cases}$$

(A) 0

(B) 1

(C) 2

(D) 4

Answer: 1

Continuity:

$y_1 = cx^2 + 2x$ is continuous on $(-\infty, 2)$ as a polynomial for all c

$y_2 = 2x + 4$ is continuous on $[2, +\infty)$ as a polynomial

We check continuity at $x = 2$

- $f(2) = 2 \cdot 2 + 4 = 8$
- $\lim_{x \to 2^-} (cx^2 + 2x) = 4c + 4$

  $\lim_{x \to 2^+} (2x + 4) = 8$

$\Rightarrow 4c + 4 = 8$

$4c = 4$

$\boxed{c = 1}$

$\Rightarrow$ for $c = 1$ f is continuous at $x = 2$

$\Rightarrow$ for $c = 1$ f is continuous for all $x \in (-\infty, +\infty)$

9. [**10 points**] For what value(s) of x does the graph of $f(x) = \frac{1}{3}x^3 - x^2 + 3$ have a horizontal tangent?

(A) 0

(B) 0 and 3

(C) 2

(D) 0 and 2

$f'(x) = x^2 - 2x$

$f'(x) = 0$ iff $x^2 - 2x = 0$

iff $x(x-2) = 0$

$\boxed{x = 0 \text{ and } x = 2}$

Answer: D

10. [**10 points**] Given the function $f(x) = x^3 - 4x^2 + 5x$, find the open interval(s) where f is concave down, i.e. where the second derivative $f''(x) < 0$.

(A) $\left(\frac{4}{3}, +\infty\right)$

(B) $\left(-\infty, \frac{4}{3}\right)$

(C) $\left(-\infty, \frac{4}{3}\right]$

(D) $\left[\frac{4}{3}, +\infty\right)$

$f'(x) = 3x^2 - 8x + 5$

$f''(x) = 6x - 8$

$f''(x) < 0$ iff $6x - 8 < 0$ iff

$x < \frac{8}{6}$

or $\boxed{x < \frac{4}{3}}$

$\Rightarrow$ f is concave down on the interval $(-\infty, 4/3)$

Answer: B

11. **[10 points]** Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.

(A) $y = 2x + 2\pi$

(B) $y = 2x$

(C) $y = -2x + 2\pi$

(D) $y = -2x$

Answer: B

$y' = 2\sin x + 2x\cos x$

$y'(\pi/2) = 2 \cdot \sin \pi/2 + 2 \cdot \frac{\pi}{2} \cdot \cos\frac{\pi}{2}$

$= 2 + \pi \cdot 0 = 2$

$\Rightarrow y - \pi = 2\left(x - \frac{\pi}{2}\right)$

$y - \pi = 2x - \pi$

$\boxed{y = 2x}$

12. **[10 points]** Find $k'(s)$ if $k(s) = \dfrac{\ln s}{s^2}$.

(A) $\dfrac{1}{2s^2}$

(B) $-\dfrac{2}{s^4}$

(C) $\dfrac{1}{s^3} + \dfrac{2\ln s}{s^3}$

(D) $\dfrac{1}{s^3} - \dfrac{2\ln s}{s^3}$

Answer: D

$k'(s) = \dfrac{\frac{1}{s} \cdot s^2 - (\ln s) \cdot 2s}{s^4}$

$= \dfrac{s - 2s\ln s}{s^4}$

$= \dfrac{1 - 2\ln s}{s^3} = \dfrac{1}{s^3} - \dfrac{2\ln s}{s^3}$

13. [**10 points**] Find $\lim_{h \to 0} \frac{|h|}{h}$.

(A) 1

(B) -1

(C) ∞

(D) The limit does not exist.

Answer: D

Since $|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$

$\Rightarrow \lim_{h \to 0^-} \frac{-h}{h} = -1$

and

$\lim_{h \to 0^+} \frac{h}{h} = 1$

$\Rightarrow$ since one sided limits are different

$\lim_{h \to 0} \frac{|h|}{h}$ does not exist

14. [**10 points**] Find the value of $\lim_{x \to \infty} \frac{x + 2}{9x^2 + 1}$.

(A) 0

(B) $\frac{1}{9}$

(C) $\frac{2}{9}$

(D) ∞

Answer: A

$$= \lim_{x \to \infty} \frac{\cancel{x^2}\left(\frac{1}{x} + \frac{2}{x^2}\right)}{\cancel{x^2}\left(9 + \frac{1}{x^2}\right)} = \frac{0}{9} = 0$$

Long-Answer Problems

Instructions: Please show all necessary work and provide full justification for each answer. Place a box around each answer.

15. [**20 points**] Let $f(x) = 2x^2 + 1$. Use the *limit definition of the derivative* to find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ 5 pts

$$= \lim_{h \to 0} \frac{2(x+h)^2 + \cancel{1} - 2x^2 - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 2x^2}{h}$$ 10 pts

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}} = 4x$$ 5 pts

Checking by using differentiation rules:

$$f'(x) = (2x^2 + 1)' = (2x^2)' + (1)' = 4x$$

16. **[20 points]** The position function of a particle is given by $s(t) = 3t^2 - t^3$, $t \geq 0$.

(a) When does the particle reach a velocity of 0 m/s? Explain the significance of this value of t.

(b) When does the particle have acceleration 0 m/s^2?

a) $v(t) = s'(t) = (3t^2 - t^3)' = 6t - 3t^2$ 5pts

$v(t) = 0$ iff $6t - 3t^2 = 0$ iff

$3t(2-t) = 0$ 3pts

iff $t = 0$ sec and $t = 2$ sec

starting point 2pts

at $t=0$ and $t=2$, particle is at rest

10pts

b) $a(t) = v'(t) = s''(t)$ (5pts)

$0 = a(t) = v'(t) = (6t - 3t^2)' = 6 - 6t = 0$ iff $\boxed{t = 1 \text{ sec}}$ 5pts

10pts

17. [**20 points**] On what interval(s) is the function $f(x) = x^3 e^x$ increasing?

$$f'(x) = 3x^2e^x + x^3e^x$$
$$= x^2e^x(3+x) > 0$$

10 pts

iff $3+x > 0$ because $x^2e^x > 0$ for all $x \neq 0$

iff $x > -3$

10 pts

$\boxed{f \text{ is increasing on } (-3,0) \cup (0,+\infty)}$

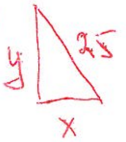
Extra Credit. [**20 points**] Choose **exactly one** of the following problems.

(i) A 25-ft ladder is leaning against a vertical wall. The bottom of the ladder is pulled horizontally away from the wall at 3 ft/sec. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 15 ft away from the wall?

(ii) Find the derivative of $f(x) = (\cos x)^x$.

(iii) Show that the function $f(x) = |x - 2|$ is continuous everywhere but not differentiable at $x = 2$. (A sketch may provide insight about this problem, but will not be considered a complete solution by itself.)

(i)



$\Rightarrow \quad x^2 + y^2 = (25)^2 \quad \Rightarrow \quad y = \sqrt{625 - 225} = 20\text{ ft}$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

when $x = 15$ ft and $\frac{dx}{dt} = 3$ ft/sec

$2 \cdot 15 \cdot 3 + 2y \cdot \frac{dy}{dt} = 0$

$90 + 2 \cdot 20 \cdot \frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-90}{40}$ ft/sec

$\Rightarrow$ The top of the ladder is sliding down the wall at the rate 9/4 ft/sec

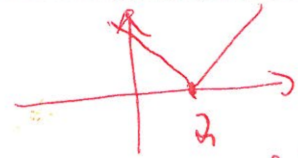
(ii) $\ln f(x) = x \ln(\cos x)$

$\frac{1}{f(x)} \cdot f'(x) = \ln(\cos x) + x\frac{1}{\cos x} \cdot (-\sin x)$

$\Rightarrow \quad f'(x) = f(x)[\ln(\cos x) - x\tan x] \quad \Rightarrow \boxed{f'(x) = (\cos x)^x[\ln(\cos x) - x\tan x]}$

(iii) $f(x) = |x-2| = \begin{cases} x-2 & \text{for } x-2 \geq 0 \\ -(x-2) & \text{for } x-2 < 0 \end{cases}$

$= \begin{cases} x-2 & \text{for } x \geq 2 \\ -x+2 & \text{for } x < 0 \end{cases}$



But $\left.\begin{matrix} f'_-(x) = -1 \\ f'_+(x) = 1 \end{matrix}\right) \Rightarrow$ left and right derivatives are diff therefore f is not differentiable at $x = 2$

Continuous everywhere because $y_1 = x - 2$ and $y_2 = -x + 2$ are continuous functions as polynomials and at $x = 2$ $f(2) = 0$

$\left.\begin{matrix} \lim_{x\to 2^-} f(x) = 0 \\ \lim_{x\to 2^+} f(x) = 0 \end{matrix}\right) \Rightarrow \lim_{x\to 0} f(x) = 0 \Bigg) \Rightarrow f(2) = \lim_{x\to 2} f(x) = 0 \Rightarrow f$ is cont. at $x=2$

$\Rightarrow f$ is cont. everywhere